Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 211 (2007) 417–435
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

The sharp Sobolev and isoperimetric inequalities split twice ☆

Jie Xiao

Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL A1C 5S7, Canada

Received 7 April 2006; accepted 24 August 2006

Available online 6 October 2006

Communicated by Erwin Lutwak

Abstract

This paper shows that each of the sharp (endpoint) Sobolev inequality and the isoperimetric inequality can be split into two sharp and stronger inequalities through either the 1-variational capacity or the 1-integral affine surface area. Furthermore, some related sharp analytic and geometric inequalities are also explored.

MSC: primary 26D, 46E; secondary 52A, 53C

Keywords: Sharpness; Analytic/geometric inequality; Volume; Surface area; Variational capacity; Integral affine surface area

1. Introduction

As is well known, the sharp (endpoint) Sobolev inequality in the Euclidean n-space $\mathbb{R}^n$, $n > 1$, says that the analytic inequality

$$\|f\|_{\frac{n}{n-1}} \leqslant \left(n\omega_n^{\frac{1}{n}}\right)^{-1} \|\nabla f\|_1 \tag{1.1}$$

holds for any $f \in C_0^1(\mathbb{R}^n)$. Here and henceforth, $C_0^1(\mathbb{R}^n)$ consists of all C^1 functions with compact support in $\mathbb{R}^n$; $\|\cdot\|_q$ ($q \geqslant 1$) is the usual L_q norm of a function on $\mathbb{R}^n$; and ω_n is the volume of the unit ball enclosed by the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

☆ Supported in part by NSERC (Canada) and Dean of Science Start-up Funds of MUN (Canada).
E-mail address: jxiao@math.mun.ca.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2006.08.006

Interestingly and importantly, Federer–Fleming (cf. [2]) and Maz'ya (cf. [12–14,16]) proved independently that (1.1) is equivalent to the following sharp isoperimetric inequality: If M is a compact domain (that is, the closure of a bounded open set) in $\mathbb{R}^n$ with C^1 boundary, then its volume $V(M)$ and surface area $S(M)$ satisfy the sharp geometric inequality

$$\big(V(M)\big)^{\frac{n-1}{n}} \leqslant \big(n\omega_n^{\frac{1}{n}}\big)^{-1} S(M). \tag{1.2}$$

Many people have been drawn to study (1.1) or/and (1.2) from a variety of directions—analytic, geometric and so on; see also [17] and [5]. Our interests in the above sharp inequalities grow out of understanding two important methods due to Maz'ya and Zhang respectively to establish (1.1) and hence (1.2).

One is regarded as Maz'ya's capacitary Sobolev inequality—see the limiting case $p = 1$ of [13, p. 109, (6) and p. 105, (7)]: If $f \in C_0^1(\mathbb{R}^n)$, then

$$\|f\|_{\frac{n}{n-1}} \leqslant \big(n\omega_n^{\frac{1}{n}}\big)^{-1} \int_0^\infty C_1\big(\big\{x \in \mathbb{R}^n\colon\ \big|f(x)\big| \geqslant t\big\}\big)\,dt \leqslant \big(n\omega_n^{\frac{1}{n}}\big)^{-1} \|\nabla f\|_1, \tag{1.3}$$

where $C_1(\cdot)$ is the 1-variational capacity defined via

$$C_1(K) = \inf\big\{\|\nabla f\|_1\colon\ f \in C_0^\infty\big(\mathbb{R}^n\big),\ f \geqslant 1_K\big\}$$

for a compact set $K \subseteq \mathbb{R}^n$; as usual, $C_0^\infty(\mathbb{R}^n)$ denotes the class of all C^∞ functions with compact support in $\mathbb{R}^n$ and 1_K means the characteristic function of $K \subseteq \mathbb{R}^n$.

The other is Zhang's affine Sobolev inequality (cf. [22, p. 194]): If $f \in C_0^1(\mathbb{R}^n)$, then

$$\|f\|_{\frac{n}{n-1}} \leqslant \left(\frac{\omega_n}{\omega_{n-1}}\right) \int_0^\infty I_1\big(\big\{x \in \mathbb{R}^n\colon\ \big|f(x)\big| \geqslant t\big\}\big)\,dt \leqslant \big(n\omega_n^{\frac{1}{n}}\big)^{-1} \|\nabla f\|_1, \tag{1.4}$$

where $I_1(\cdot)$ is the 1-integral affine surface area determined through a spherical projection—more precisely, for a compact domain $K \subseteq \mathbb{R}^n$ with the C^1 boundary ∂K; the surface area element ds_K; and the exterior unit normal vector ν, the mapping

$$f \mapsto L_K(f) = \int_{\partial K} f\big(\nu(x)\big)\,ds_K(x)$$

produces a bounded linear functional on $C(\mathbb{S}^{n-1})$ which comprises all continuous functions on $\mathbb{S}^{n-1}$. So, there is a finite positive Borel measure μ_K on $\mathbb{S}^{n-1}$, called the (classic) surface area measure associated to K, such that

$$L_K(f) = \int_{\mathbb{S}^{n-1}} f\,d\mu_K, \quad f \in C\big(\mathbb{S}^{n-1}\big).$$

Using the projection function of K on $\mathbb{S}^{n-1}$

$$v(K,u)=2^{-1}\int_{\partial K}\big|\langle u,\nu(x)\rangle\big|\,ds_K(x)=2^{-1}\int_{\mathbb{S}^{n-1}}\big|\langle u,v\rangle\big|\,d\mu_K(v),\quad u\in\mathbb{S}^{n-1},$$

where $\langle\cdot,\cdot\rangle$ means the usual inner product of two points in $\mathbb{R}^n$, we define the 1-integral affine surface area of K as

$$I_1(K)=\left(n^{-1}\int_{\mathbb{S}^{n-1}}\big(v(K,u)\big)^{-n}\,du\right)^{-\frac{1}{n}}.$$

In this paper, we figure out that each of (1.1) and (1.2) can be further split into two sharp stronger inequalities by means of either the 1-variational capacity or the 1-integral affine surface area.

The first two splitting inequalities are obtained via the 1-variational capacity.

Theorem 1.1. *Let $n>1$. Then*

(i) *The analytic inequality*

$$\|f\|_{\frac{n}{n-1}}\leqslant\big(n\omega_n^{\frac{1}{n}}\big)^{-1}\left(\int_0^\infty\Big(C_1\big(\{x\in\mathbb{R}^n:\ |f(x)|\geqslant t\}\big)\Big)^{\frac{n}{n-1}}\,dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\tag{1.5}$$

for any Lebesgue measurable function f with compact support in $\mathbb{R}^n$, is equivalent to, the geometric inequality

$$\big(V(M)\big)^{\frac{n-1}{n}}\leqslant\big(n\omega_n^{\frac{1}{n}}\big)^{-1}C_1(M)\tag{1.6}$$

for any compact domain M in $\mathbb{R}^n$.

(ii) *The inequalities* (1.5) *and* (1.6) *are true and sharp.*

Theorem 1.2. *Let $n>1$. Then*

(i) *The analytic inequality*

$$\left(\int_0^\infty\Big(C_1\big(\{x\in\mathbb{R}^n:\ |f(x)|\geqslant t\}\big)\Big)^{\frac{n}{n-1}}\,dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\leqslant\|\nabla f\|_1,\quad f\in C_0^1\big(\mathbb{R}^n\big),\tag{1.7}$$

is equivalent to, the geometric inequality

$$C_1(M)\leqslant S(M)\tag{1.8}$$

for any compact domain M in $\mathbb{R}^n$ with C^1 boundary.

(ii) *The inequalities* (1.7) *and* (1.8) *are true and sharp.*

Clearly, (1.5) and (1.7) are stronger than (1.1), but also (1.6) and (1.8) are stronger than (1.2). After the fashion of the 1-variational capacity, we can also establish next two splitting inequalities involving the 1-integral affine surface area.

Theorem 1.3. *Let $n > 1$. Then*

(i) *The analytic inequality*

$$\|f\|_{\frac{n}{n-1}} \leqslant \left(\frac{\omega_n}{\omega_{n-1}}\right)\left(\int_0^\infty \left(I_1\left(\left\{x \in \mathbb{R}^n:\ \left|f(x)\right| \geqslant t\right\}\right)\right)^{\frac{n}{n-1}} dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}, \quad f \in C_0^1\left(\mathbb{R}^n\right), \tag{1.9}$$

is equivalent to, the geometric inequality

$$\left(V(M)\right)^{\frac{n-1}{n}} \leqslant \left(\frac{\omega_n}{\omega_{n-1}}\right) I_1(M) \tag{1.10}$$

for any compact domain M in $\mathbb{R}^n$ with C^1 boundary.

(ii) *The inequalities* (1.9) *and* (1.10) *are true and sharp.*

Theorem 1.4. *Let $n > 1$. Then*

(i) *The analytic inequality*

$$\left(\int_0^\infty \left(I_1\left(\left\{x \in \mathbb{R}^n:\ \left|f(x)\right| \geqslant t\right\}\right)\right)^{\frac{n}{n-1}} dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leqslant \left(\frac{\omega_{n-1}}{n\omega_n^{1+\frac{1}{n}}}\right)\|\nabla f\|_1, \quad f \in C_0^1\left(\mathbb{R}^n\right), \tag{1.11}$$

is equivalent to, the geometric inequality

$$I_1(M) \leqslant \left(\frac{\omega_{n-1}}{n\omega_n^{1+\frac{1}{n}}}\right) S(M) \tag{1.12}$$

for any compact domain M in $\mathbb{R}^n$ with C^1 boundary.

(ii) *The inequalities* (1.11) *and* (1.12) *are true and sharp.*

Obviously, the inequalities (1.9) and (1.11) imply (1.1); at the same time, the inequalities (1.10) and (1.12) derive (1.2).

Perhaps, it is worth emphasizing that Theorems 1.1–1.2 and Theorems 1.3–1.4 appear to be surprisingly similar for the format although the following comparison shows that the 1-variational capacity and the 1-integral affine surface area behave differently.

Theorem 1.5. *Let $n > 1$. Then*

(i) *The analytic inequality*

$$\left(\int_0^\infty \left(I_1\left(\left\{x \in \mathbb{R}^n\colon \left|f(x)\right| \geqslant t\right\}\right)\right)^{\frac{n}{n-1}} dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}$$

$$\leqslant \left(\frac{\omega_{n-1}}{n\omega_n^{1+\frac{1}{n}}}\right)\left(\int_0^\infty \left(C_1\left(\left\{x \in \mathbb{R}^n\colon \left|f(x)\right| \geqslant t\right\}\right)\right)^{\frac{n}{n-1}} dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}, \quad f \in C_0^1(\mathbb{R}^n), \tag{1.13}$$

is equivalent to, the geometric inequality

$$I_1(M) \leqslant \left(\frac{\omega_{n-1}}{n\omega_n^{1+\frac{1}{n}}}\right)C_1(M) \tag{1.14}$$

for any compact domain M in $\mathbb{R}^n$ with C^1 boundary.

(ii) *The inequalities* (1.13) *and* (1.14) *are true and sharp.*

The proofs of the above five theorems are arranged in the forthcoming section. Here it should be pointed out that (1.6) and (1.7) may be respectively viewed as certain formal limits of Maz'ya's estimates in [13, p. 105, (7)] and [15, Proposition 1]; meanwhile (1.10) and (1.12) are due to Zhang (cf. [22, p. 191] and references therein), but otherwise the results in the theorems are new. Our principal task of this note is to prove the equivalences stated in Theorems 1.1–1.5 and the sharpnesses of the corresponding inequalities. Moreover, our techniques can used to prove that the left/right-hand side inequality of (1.3) is equivalent to (1.6)/(1.8), the left/right-hand side inequality of (1.4) amounts to (1.10)/(1.12), and the sharp geometric inequality (1.14) is equivalent to the sharp analytic inequality

$$\int_0^\infty I_1\left(\left\{x \in \mathbb{R}^n\colon \left|f(x)\right| \geqslant t\right\}\right) dt$$

$$\leqslant \left(\frac{\omega_{n-1}}{n\omega_n^{1+\frac{1}{n}}}\right)\int_0^\infty C_1\left(\left\{x \in \mathbb{R}^n\colon \left|f(x)\right| \geqslant t\right\}\right) dt, \quad f \in C_0^1(\mathbb{R}^n).$$

This paper ends up with the third section in which we address ourselves to some possible extensions of these five theorems from $p = 1$ to $p \in (1, n)$.

2. Proofs

In the succeeding demonstrations of Theorems 1.1–1.5 and the results of the final section, we will always adopt two short notations:

$$\Omega_t(f) = \left\{x \in \mathbb{R}^n\colon \left|f(x)\right| \geqslant t\right\}$$

and

$$\partial\Omega_t(f) = \left\{x \in \mathbb{R}^n\colon \left|f(x)\right| = t\right\}$$

for a function f defined on $\mathbb{R}^n$ and a number $t > 0$.

Proof of Theorem 1.1. (i) Given a compact domain $M \subseteq \mathbb{R}^n$, let $f = 1_M$. Then

$$\|f\|_{\frac{n}{n-1}} = \big(V(M)\big)^{\frac{n-1}{n}}$$

and

$$\Omega_t(f) = \begin{cases} M, & \text{if } t \in (0,1], \\ \emptyset, & \text{if } t \in (1,\infty). \end{cases}$$

Hence

$$\begin{aligned} &\int_0^\infty \big(C_1\big(\Omega_t(f)\big)\big)^{\frac{n}{n-1}}\, dt^{\frac{n}{n-1}} \\ &= \int_0^1 \big(C_1\big(\Omega_t(f)\big)\big)^{\frac{n}{n-1}}\, dt^{\frac{n}{n-1}} + \int_1^\infty \big(C_1\big(\Omega_t(f)\big)\big)^{\frac{n}{n-1}}\, dt^{\frac{n}{n-1}} \\ &= \big(C_1(M)\big)^{\frac{n}{n-1}} \end{aligned}$$

and so (1.5) implies (1.6).

Conversely, we verify that (1.6) implies (1.5). Suppose (1.6) holds for any compact domain in $\mathbb{R}^n$. For $t > 0$ and f, a Lebesgue measurable function with compact support in $\mathbb{R}^n$, we use the definition of Lebesgue $\frac{n}{n-1}$–integral and (1.6) to get

$$\|f\|_{\frac{n}{n-1}}^{\frac{n}{n-1}} = \int_0^\infty V\big(\Omega_t(f)\big)\, dt^{\frac{n}{n-1}} \leqslant \big(n\omega_n^{\frac{1}{n}}\big)^{-\frac{n}{n-1}} \int_0^\infty \big(C_1\big(\Omega_t(f)\big)\big)^{\frac{n}{n-1}}\, dt^{\frac{n}{n-1}},$$

thereby yielding (1.5).

(ii) Thanks to the equivalence between (1.5) and (1.6), it suffices to prove that (1.6) is valid and sharp. In fact, (1.6) follows from an application of the definition of $C_1(\cdot)$ to (1.1). The sharpness of (1.6) can be seen from evaluating $V(M)$ and $C_1(M)$ for $M = \overline{B(x,r)}$—the closed ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$; see also the argument for Theorem 1.2(ii) below. □

Proof of Theorem 1.2. (i) That (1.7) implies (1.8) will be deduced by a smooth approximation of the characteristic function of a compact domain M. As a matter of fact, for $\epsilon > 0$ and $M \subseteq \mathbb{R}^n$, a compact domain with C^1 boundary, let

$$M_\epsilon = \big\{x \in \mathbb{R}^n\colon\ 0 < \operatorname{dist}(x, M) < \epsilon\big\}$$

(where $\operatorname{dist}(x, M)$ stands for the Euclidean distance of x to M) and

$$f_\epsilon(x) = \begin{cases} 1 - \epsilon^{-1} \operatorname{dist}(x, M), & \operatorname{dist}(x, M) < \epsilon, \\ 0, & \operatorname{dist}(x, M) \geqslant \epsilon. \end{cases}$$

Note that this $C_0^1(\mathbb{R}^n)$ function f_ϵ approaches to the characteristic function 1_M of M as $\epsilon \to 0$. When $x \in M_\epsilon$ with ϵ being very small, we can choose a unique point $y \in \partial M$ such that $\operatorname{dist}(x, M) = |y - x|$. With $\nu(y) = |y - x|^{-1}(y - x)$ and $\overline{M_\epsilon}$, the closure of M_ϵ, we further have

$$\nabla f_\epsilon(x) = \begin{cases} \epsilon^{-1}\nu(y), & x \in M_\epsilon, \\ 0, & x \notin \overline{M_\epsilon}. \end{cases}$$

This implies in turn that $\|\nabla f_\epsilon\|_1 \to S(M)$ as $\epsilon \to 0$. Meantime, since

$$M = \{x \in \mathbb{R}^n \colon \operatorname{dist}(x, M) = 0\},$$

we conclude that if (1.7) holds then

$$C_1(M) \leqslant \left(\int_0^1 \big(C_1\big(\Omega_t(f_\epsilon)\big)\big)^{\frac{n}{n-1}}\, dt^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leqslant \|\nabla f_\epsilon\|_1,$$

whence giving (1.8).

Suppose (1.8) is true for any compact domain in $\mathbb{R}^n$ with C^1 boundary. Now, given any $C_0^1(\mathbb{R}^n)$ function f, we have that for almost all $t > 0$ the boundary $\partial\Omega_t(f)$ is a C^1 submanifold with the nonzero normal vector ∇f. Noticing that $S(\Omega_t(f))$ decreases in t, we obtain

$$\begin{aligned} t^{\frac{1}{n-1}}\big(S\big(\Omega_t(f)\big)\big)^{\frac{n}{n-1}} &= \big(tS\big(\Omega_t(f)\big)\big)^{\frac{1}{n-1}} S\big(\Omega_t(f)\big) \\ &\leqslant \left(\int_0^t S\big(\Omega_r(f)\big)\, dr \right)^{\frac{1}{n-1}} S\big(\Omega_t(f)\big) \\ &= \left(\frac{n-1}{n}\right) \frac{d}{dt}\left(\left(\int_0^t S\big(\Omega_r(f)\big)\, dr \right)^{\frac{n}{n-1}} \right). \end{aligned}$$

Because $ds_{\Omega_t(f)}\, dt$ is equal to $|\nabla f(x)|\, dx$, (1.8), along with the last estimate, leads to

$$\begin{aligned} \int_0^\infty \big(C_1\big(\Omega_t(f)\big)\big)^{\frac{n}{n-1}}\, dt^{\frac{n}{n-1}} &\leqslant \int_0^\infty \big(S\big(\Omega_t(f)\big)\big)^{\frac{n}{n-1}}\, dt^{\frac{n}{n-1}} \\ &= \left(\frac{n}{n-1}\right) \int_0^\infty t^{\frac{1}{n-1}} \big(S\big(\Omega_t(f)\big)\big)^{\frac{n}{n-1}}\, dt \\ &\leqslant \int_0^\infty \frac{d}{dt}\left(\left(\int_0^t S\big(\Omega_r(f)\big)\, dr \right)^{\frac{n}{n-1}} \right) dt \end{aligned}$$

$$= \left(\int_0^\infty S\big(\Omega_t(f)\big)\, dt \right)^{\frac{n}{n-1}}$$

$$= \left(\int_0^\infty \int_{\partial \Omega_t(f)} ds_{\Omega_t(f)}\, dt \right)^{\frac{n}{n-1}}$$

$$= \|\nabla f\|_1^{\frac{n}{n-1}}.$$

Thus (1.7) follows.

(ii) Due to the equivalence between (1.7) and (1.8), it is enough to check that (1.8) is not only valid for any compact domain in $\mathbb{R}^n$, but also sharp. In fact, (1.8) follows directly from either [13, p. 107, Lemma] or the validity of (1.7)—a special case of the Maz'ya capacitary inequality in [15, Proposition 1] plus the argument for (1.7) $\Rightarrow$ (1.8). To see its sharpness, we calculate $C_1(M)$ for $M = \overline{B(x,r)}$. If $f \in C_0^\infty(\mathbb{R}^n)$ satisfies $f \geqslant 1_M$, and if $z \in \partial M$ then

$$1 \leqslant \int_r^\infty \big|\nabla f(x+tz)\big|\, dt$$

and thus by integrating with respect to the spherical measure,

$$n\omega_n \leqslant \int_{\mathbb{R}^n \setminus B(x,r)} |y-x|^{1-n} \big|\nabla f(y)\big|\, dy \leqslant r^{1-n}\|\nabla f\|_1.$$

This yields

$$S\big(\overline{B(x,r)}\big) = n\omega_n r^{n-1} \leqslant C_1\big(\overline{B(x,r)}\big).$$

Furthermore, this estimate and (1.8) produce

$$C_1\big(\overline{B(x,r)}\big) = S\big(\overline{B(x,r)}\big).$$

Accordingly, the equality in (1.8) is valid for $M = \overline{B(x,r)}$. □

Proof of Theorem 1.3. (i) Suppose (1.10) is valid for any compact domain in $\mathbb{R}^n$ with C^1 boundary. For any $C_0^1(\mathbb{R}^n)$ function f, we obtain via applying (1.10) to $\Omega_t(f)$ that

$$\|f\|_{\frac{n}{n-1}} = \left(\int_0^\infty V\big(\Omega_t(f)\big)\, dt^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}$$

$$\leqslant \left(\frac{\omega_n}{\omega_{n-1}} \right) \left(\int_0^\infty \big(I_1\big(\Omega_t(f)\big)\big)^{\frac{n}{n-1}}\, dt^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}},$$

and so that (1.9) holds.

Conversely, suppose (1.9) is true. Now for any $\epsilon \in (0,1)$ and compact domain $M \subseteq \mathbb{R}^n$ with C^1 boundary, let M_ϵ, $\overline{M_\epsilon}$, f_ϵ and $\nu(y)$ be the same as in the proof of that (1.7) implies (1.8). It is clear that

$$\Omega_t(f_\epsilon) \subseteq \begin{cases} \emptyset, & t > 1, \\ \overline{M_\epsilon}, & 0 < t \leqslant 1. \end{cases}$$

If $t = \operatorname{dist}(x, M) \in (0, \epsilon)$ and $u \in \mathbb{S}^{n-1}$, then $dx = ds_M\, dt + o(\Delta t)$ and hence (cf. [22, p. 195])

$$\epsilon^{-1} \int_{M_\epsilon} \big|\langle u, \nu(y)\rangle\big|\, dx \to \int_{\partial M} \big|\langle u, \nu(y)\rangle\big|\, ds_M(y) = 2v(M,u) \quad \text{as } \epsilon \to 0,$$

which actually means

$$\lim_{\epsilon \to 0} v(\overline{M_\epsilon}, u) = v(M, u).$$

Note that $\overline{M_\epsilon}$ shrinks to M as ϵ decreases to 0, but also the following formula

$$v(K,u) = 2^{-1} \int \#(K \cap \ell_u)\, d\ell_u, \quad u \in \mathbb{S}^{n-1},$$

holds for any compact domain K with C^1 boundary, where ℓ_u and $d\ell_u$ are respectively a line parallel to the unit vector u and the volume element of the subspace $u^\perp$ orthogonal to u. So, the definition of $I_1(\cdot)$ yields that $I_1(\overline{M_\epsilon})$ decreases to $I_1(M)$ as ϵ decreases to 0. Applying (1.9) to f_ϵ and noting $I_1(\emptyset) = 0$, we derive

$$\big(V(M)\big)^{\frac{n-1}{n}} \leftarrow \|f_\epsilon\|_{\frac{n}{n-1}} \leqslant \left(\frac{\omega_n}{\omega_{n-1}}\right) \left(\int_0^1 \big(I_1(\overline{M_\epsilon})\big)^{\frac{n}{n-1}}\, dt^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \to \left(\frac{\omega_n}{\omega_{n-1}}\right) I_1(M)$$

as $\epsilon \to 0$. Therefore, (1.10) follows.

(ii) Since the truth of (1.10) has actually been checked by Zhang in [22]. So the validity of (1.9) follows from the equivalence between (1.9) and (1.10). Moreover, the sharpness of (1.10) follows from the known fact (cf. [22, p. 191, Lemma 3.5]) that the equality in (1.10) takes place when M is an ellipsoid in $\mathbb{R}^n$. Accordingly, (1.9) is sharp, too. □

Proof of Theorem 1.4. (i) Suppose (1.11) is valid. Given $\epsilon \in (0,1)$ and $M \subseteq \mathbb{R}^n$, a compact domain with C^1 boundary, let again M_ϵ, $\overline{M_\epsilon}$, f_ϵ and $\nu(y)$ be as before. Since $x \in M$ implies $\operatorname{dist}(x, M) = 0$, we conclude that $M \subseteq \Omega_t(f_\epsilon)$ for any $t \in (0,1]$. This inclusion, together with the definition of $I_1(\cdot)$ and the just-mentioned formula of $v(\cdot,\cdot)$, deduces

$$\big(I_1\big(\Omega_t(f_\epsilon)\big)\big)^{\frac{n}{n-1}} \geqslant \big(I_1(M)\big)^{\frac{n}{n-1}}, \quad t \in (0,1].$$

Hence (1.11) (applied to f_ϵ with ϵ being very small) derives

$$I_1(M) \leqslant \left(\int_0^1 \bigl(I\bigl(\Omega_t(f_\epsilon)\bigr)\bigr)^{\frac{n}{n-1}}\, dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leqslant \left(\frac{\omega_{n-1}}{n\omega_n^{1+\frac{1}{n}}}\right)\|\nabla f_\epsilon\|_1.$$

This inequality, along with the limit $\lim_{\epsilon\to 0}\|\nabla f_\epsilon\|_1 = S(M)$, gives (1.12).

Conversely, if (1.12) holds for any compact domain in $\mathbb{R}^n$ with C^1 boundary, then for f, a $C_0^1(\mathbb{R}^n)$ function, we use the argument for (1.8) $\Rightarrow$ (1.7) to achieve

$$\begin{aligned}\left(\int_0^\infty \bigl(I_1\bigl(\Omega_t(f)\bigr)\bigr)^{\frac{n}{n-1}}\, dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} &\leqslant \left(\frac{\omega_{n-1}}{n\omega_n^{1+\frac{1}{n}}}\right)\left(\int_0^\infty \bigl(S\bigl(\Omega_t(f)\bigr)\bigr)^{\frac{n}{n-1}}\, dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\\ &\leqslant \left(\frac{\omega_{n-1}}{n\omega_n^{1+\frac{1}{n}}}\right)\|\nabla f\|_1.\end{aligned}$$

Namely, (1.11) is valid.

(ii) The truth of (1.12) has been verified by Zhang in [22, p. 191]. Accordingly, the truth of (1.11) follows from the equivalence between (1.11) and (1.12). To see the sharpnesses of (1.11) and (1.12), it is enough to do this for (1.12). In fact, noticing that if $M = \overline{B(x,r)}$ then the equality in (1.10) occurs, we find

$$\bigl(I_1(M)\bigr)^{-n} = \left(\frac{\omega_n}{\omega_{n-1}^n}\right)r^{(1-n)n} = \left(\frac{n\omega_n^{1+\frac{1}{n}}}{\omega_{n-1}}\right)^n \bigl(S(M)\bigr)^{-n}.$$

In other words, the equality in (1.12) is attained. $\square$

Proof of Theorem 1.5. (i) Clearly, it suffices to show that (1.13) implies (1.14). To the end, suppose (1.13) holds. Now, for any compact domain $M \subseteq \mathbb{R}^n$ with C^1 boundary and $\epsilon \in (0,1)$ we consider the preceding f_ϵ and M_ϵ once again. Noticing the above-established facts:

$$\Omega_t(f_\epsilon) = \emptyset, \quad t > 1,$$

and

$$M \subseteq \Omega_t(f_\epsilon) \subseteq \overline{M_\epsilon}, \quad t \in (0,1],$$

we get by (1.13) that

$$\begin{aligned}I_1(M) &\leqslant \left(\int_0^1 \bigl(I_1\bigl(\Omega_t(f_\epsilon)\bigr)\bigr)^{\frac{n}{n-1}}\, dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\\ &= \left(\int_0^\infty \bigl(I_1\bigl(\Omega_t(f_\epsilon)\bigr)\bigr)^{\frac{n}{n-1}}\, dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\end{aligned}$$

$$\leqslant \left(\frac{\omega_{n-1}}{n\omega_n^{1+\frac{1}{n}}}\right)\left(\int_0^\infty \left(C_1\left(\Omega_t(f_\epsilon)\right)\right)^{\frac{n}{n-1}}\,dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}$$

$$= \left(\frac{\omega_{n-1}}{n\omega_n^{1+\frac{1}{n}}}\right)\left(\int_0^1 \left(C_1\left(\Omega_t(f_\epsilon)\right)\right)^{\frac{n}{n-1}}\,dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}$$

$$\leqslant \left(\frac{\omega_{n-1}}{n\omega_n^{1+\frac{1}{n}}}\right)C_1(\overline{M_\epsilon})$$

$$\to \left(\frac{\omega_{n-1}}{n\omega_n^{1+\frac{1}{n}}}\right)C_1(M) \quad \text{as } \epsilon \to 0.$$

Therefore, (1.14) holds.

(ii) Given a compact domain M in $\mathbb{R}^n$ with C^1 boundary. If $f \in C_0^\infty(\mathbb{R}^n)$ and $f \geqslant 1_M$, then from (1.11) it follows that

$$I_1(M) \leqslant \left(\int_0^1 \left(I_1\left(\Omega_t(f)\right)\right)^{\frac{n}{n-1}}\,dt^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leqslant \left(\frac{\omega_{n-1}}{n\omega_n^{1+\frac{1}{n}}}\right)\|\nabla f\|_1.$$

Accordingly, this estimate, together with taking the infimum over all these functions f, derives (1.14); that is,

$$I_1(M) \leqslant \left(\frac{\omega_{n-1}}{n\omega_n^{1+\frac{1}{n}}}\right)C_1(M),$$

for which the equality occurs when M is any closed ball in $\mathbb{R}^n$. This, along with the equivalence between (1.13) and (1.14), implies (1.13) and its sharpness. $\square$

3. Beyond

Recalling the well-known sharp $1 < p$-Sobolev inequality (cf. [1,18,20])

$$\|f\|_{\frac{pn}{n-p}} \leqslant \left(\frac{(\frac{p-1}{n-p})^{1-\frac{1}{p}}}{n^{\frac{1}{p}}\omega_n^{\frac{1}{n}}}\right)\psi_{p,n}^{\frac{1}{n}}\|\nabla f\|_p, \quad f \in C_0^1\left(\mathbb{R}^n\right), \tag{3.1}$$

where

$$\psi_{p,n} = \frac{\Gamma(n)}{\Gamma(\frac{n}{p})\Gamma(1+n-\frac{n}{p})}$$

for which $\Gamma(\cdot)$ is the usual gamma function, we naturally ask a follow-up question:

Can Theorems 1.1–1.5 be extended to $p > 1$?

Concerning this question associated with Theorems 1.1–1.2, we can employ the $1 < p$-variational capacity of a compact subset K of $\mathbb{R}^n$

$$\mathrm{C}_p(K) = \inf\bigl\{\|\nabla f\|_p^p\colon\ f \in C_0^\infty\bigl(\mathbb{R}^n\bigr),\ f \geqslant 1_K\bigr\},$$

to get a partial answer.

Theorem 3.1. *Let $1 < p < n$. Then*

(i) *The analytic inequality*

$$\|f\|_{\frac{pn}{n-p}} \leqslant \left(\frac{(\frac{p-1}{n-p})^{1-\frac{1}{p}}}{n^{\frac{1}{p}}\omega_n^{\frac{1}{n}}}\right)\left(\int_0^\infty \bigl(C_p\bigl(\bigl\{x \in \mathbb{R}^n\colon\ \bigl|f(x)\bigr| \geqslant t\bigr\}\bigr)\bigr)^{\frac{n}{n-p}}\, dt^{\frac{pn}{n-p}}\right)^{\frac{n-p}{pn}} \tag{3.2}$$

for any Lebesgue measurable function f with compact support in $\mathbb{R}^n$, is equivalent to, the geometric inequality

$$\bigl(V(M)\bigr)^{\frac{n-p}{pn}} \leqslant \left(\frac{(\frac{p-1}{n-p})^{1-\frac{1}{p}}}{n^{\frac{1}{p}}\omega_n^{\frac{1}{n}}}\right)\bigl(C_p(M)\bigr)^{\frac{1}{p}} \tag{3.3}$$

for any compact domain M in $\mathbb{R}^n$. The two inequalities are true.

(ii) *The analytic inequality*

$$\left(\int_0^\infty \bigl(C_p\bigl(\bigl\{x \in \mathbb{R}^n\colon\ \bigl|f(x)\bigr| \geqslant t\bigr\}\bigr)\bigr)^{\frac{n}{n-p}}\, dt^{\frac{pn}{n-p}}\right)^{\frac{n-p}{pn}} \leqslant \psi_{p,n}^{\frac{1}{n}}\|\nabla f\|_p \tag{3.4}$$

holds for any $f \in C_0^1(\mathbb{R}^n)$.

(iii) *The inequalities* (3.2) *and* (3.4) *are sharp and stronger than* (3.1).

Proof. Note that the inequalities (3.3) (as a classical isocapacitary inequality) and (3.4) are established by Maz'ya—see [13, p. 105, (7)] and [14, Proposition 1], but also (3.2) and (3.4) imply evidently (3.1). Thus, the verification of Theorem 3.1 will be completed by checking the assertion (i) and its sharpness.

Suppose (3.3) holds. For any Lebesgue measurable function f with compact support in $\mathbb{R}^n$, we then have

$$\|f\|_{\frac{pn}{n-p}}^{\frac{pn}{n-p}} = \int_0^\infty V(\Omega_t)\, dt^{\frac{pn}{n-p}} \leqslant \left(\frac{(\frac{p-1}{n-p})^{1-\frac{1}{p}}}{n^{\frac{1}{p}}\omega_n^{\frac{1}{n}}}\right)^{\frac{pn}{n-p}} \int_0^\infty \bigl(C_p(\Omega_t)\bigr)^{\frac{n}{n-p}}\, dt^{\frac{pn}{n-p}},$$

and (3.2) at once.

Conversely, if (3.2) is true, then by taking $f = 1_M$ there we get (3.3) right away. Whenever $M = \overline{B(x,r)}$, it turns out from the capacitary calculation of a closed ball done in [13, p. 106] that

$$C_p(M) = n\omega_n\left(\frac{n-p}{p-1}\right)^{p-1} r^{n-p}$$

and then the equality of (3.3) takes place. This derives the sharpness of (3.3) as well as (3.2). □

Remark 3.2. Keeping the notations used in Theorem 3.1, we make two comments:

(i) Based on the estimate

$$\|f\|_{\frac{pn}{n-p}} \leqslant \int_0^\infty \left(V\left(\left\{x \in \mathbb{R}^n \colon \left|f(x)\right| \geqslant t\right\}\right)\right)^{\frac{n-p}{pn}} dt$$

for any Lebesgue measurable function f with compact support in $\mathbb{R}^n$, we can read that the following integral:

$$\int_0^\infty \left(C_p\left(\left\{x \in \mathbb{R}^n \colon \left|f(x)\right| \geqslant t\right\}\right)\right)^{\frac{1}{p}} dt$$

may serve in place of the right-hand functional in (3.2)

$$\left(\int_0^\infty \left(C_p\left(\left\{x \in \mathbb{R}^n \colon \left|f(x)\right| \geqslant t\right\}\right)\right)^{\frac{n}{n-p}} dt^{\frac{pn}{n-p}}\right)^{\frac{n-p}{pn}},$$

but we cannot examine if the last but one integral may still substitute for the left-hand part of (3.4).

(ii) It is unknown whether (3.4) and (3.1) have the sharp geometric forms similar to (1.8) and (1.2), respectively. Maybe, introducing a p-surface area (generalizing the usual surface area, i.e., the case $p = 1$) would be useful for handling this issue.

Next, when settling the question applied to Theorems 1.3 and 1.4, we turn our course to Lutwak's L_p Brunn–Minkowski–Firey theory (cf. [6,7]) developing Firey's theorem in [3]: Under $p > 1$, the sum of the pth powers of the support functions of two convex bodies in $\mathbb{R}^n$ containing the origin is also the pth power of a support function. For our purpose, we select some of very basic facts on convex bodies—see also [4,19,21] for more information.

A convex body in $\mathbb{R}^n$ is a compact convex set with nonempty interior. Following [6,7], we use $\mathcal{K}_0^n$ as the class of all convex bodies that contain the origin in their interior. With this convention, we can say that each $K \in \mathcal{K}_0^n$ is uniquely determined by its support function

$$h(K,x) = h_K(x) = \max\left\{\langle x, y\rangle \colon y \in K\right\}, \quad x \in \mathbb{R}^n.$$

Given two convex bodies K and L in $\mathcal{K}_0^n$. For $p \geqslant 1$ and $\epsilon > 0$ let $K +_p \epsilon L$ be the Minkowski–Firey L_p combination which is the convex body in $\mathcal{K}_0^n$ with support function

$$\big(h(K +_p \epsilon L, \cdot)\big)^p = \big(h(K, \cdot)\big)^p + \epsilon\big(h(L, \cdot)\big)^p.$$

Particularly, the L_1 combination can be written as

$$K + \epsilon L = \big\{y + \epsilon z \in \mathbb{R}^n \colon\ y \in K,\ z \in L\big\}.$$

Now, the L_p-mixed volume of K and L is defined by

$$V_p(K, L) = pn^{-1} \lim_{\epsilon \to 0} \epsilon^{-1}\big(V(K +_p \epsilon L) - V(K)\big).$$

Interestingly, we have

$$V_p(M, M) = V(M), \quad M \in \mathcal{K}_0^n.$$

Even more interestingly, as proved in Lutwak [6], there is a unique finite positive Borel measure $ds_{p,K}(\cdot)$ on $\mathbb{S}^{n-1}$ such that

$$V_p(K, M) = n^{-1} \int\limits_{\mathbb{S}^{n-1}} \big(h_M(u)\big)^p \, ds_{p,K}(u), \quad M \in \mathcal{K}_0^n.$$

Following [8] and [9] of Lutwak–Yang–Zhang, we call this measure the L_p-surface area measure associated to K. In particular, $s_{1,K}$ is the classic surface area measure μ_K on $\mathbb{S}^{n-1}$. One more property of this measure says that $s_{p,K}$ is absolutely continuous with respect to $s_{1,K}$ and the corresponding Radon–Nikodym derivative is given by

$$\frac{ds_{p,K}}{ds_{1,K}}(u) = \frac{ds_{p,K}}{d\mu_K}(u) = h_K^{1-p}(u), \quad u \in \mathbb{S}^{n-1}.$$

Especially, if ∂K is C^2 with positive Gauss curvature then the Radon–Nikodym derivative of the classical surface area measure with respect to the Lebesgue measure on $\mathbb{S}^{n-1}$ equals the reciprocal of the Gauss curvature of ∂K.

The foregoing L_p-mixed volume $V_p(K, L)$ can be generalized to the case where K is a compact convex set having the origin as one of its interior points and L is a compact convex set containing the origin in its relative interior.

On the other hand, for $u \in \mathbb{S}^{n-1}$ denote by $\bar{u}$ the closed line segment between $-u/2$ and $u/2$. Then it follows that

$$V_p(K, \bar{u}) = 2^{-p} n^{-1} \int\limits_{\mathbb{S}^{n-1}} \big|\langle u, v\rangle\big|^p \, ds_{p,K}(v), \quad K \in \mathcal{K}_0^n.$$

This formula induces a natural generalization of $I_1(\cdot)$; that is, the p-integral affine surface area of a convex body $K \in \mathcal{K}_0^n$ as follows:

$$I_p(K) = n^{1+\frac{p}{n}}\left(\int\limits_{\mathbb{S}^{n-1}} \left(V_p(K,\bar{u})\right)^{-\frac{n}{p}}\,du\right)^{-\frac{p}{n}}$$
$$= \left(\int\limits_{\mathbb{S}^{n-1}} \left(\left(\frac{n^{\frac{p}{n}}}{2^p}\right)\int\limits_{\mathbb{S}^{n-1}} \left|\langle v,u\rangle\right|^p\,ds_{p,K}(v)\right)^{-\frac{n}{p}}\,du\right)^{-\frac{p}{n}}.$$

Given $f \in C_0^\infty(\mathbb{R}^n)$, $t > 0$, and M—a compact convex subset of $\mathbb{R}^n$ containing the origin in its relative interior. Let

$$V_p(f,t,M) = n^{-1}\int\limits_{\partial\Omega_t(f)} \left(h_M\left(\frac{\nabla f(x)}{|\nabla f(x)|}\right)\right)^p \left|\nabla f(x)\right|^{p-1}\,ds_{\Omega_t(f)}(x).$$

Then (cf. [9, (3.2)])

$$V_p(f,t,\bar{u}) = 2^{-p}n^{-1}\int\limits_{\partial\Omega_t(f)} \left|\langle u,\nabla f(x)\rangle\right|^p \left|\nabla f(x)\right|^{-1}\,ds_{\Omega_t(f)}(x), \quad u \in \mathbb{S}^{n-1}.$$

Furthermore, according to the Lutwak–Yang–Zhang lemma [9, Lemma 3.2], for almost every $t > 0$ there is an origin-symmetric convex body $K_t(f) \in \mathcal{K}_0^n$ such that

$$V\big(K_t(f)\big) = V_p\big(f,t,K_t(f)\big)$$

and

$$V_p\big(K_t(f),\bar{u}\big) = V_p(f,t,\bar{u}), \quad u \in \mathbb{S}^{n-1}.$$

With the help of the above-introduced notations and the following abridgment:

$$\phi_{p,n} = \frac{\Gamma(\frac{n+p}{2})}{\Gamma(1+\frac{n}{2})\Gamma(\frac{1+p}{2})},$$

we can state the following assertion.

Theorem 3.3. *Let $1 < p < n$. Then*

(i) *The analytic inequality*

$$\|f\|_{\frac{pn}{n-p}} \leqslant \left(\frac{2^{1-\frac{1}{p}}\sqrt{\pi}^{\frac{1}{p}}\psi_{p,n}^{\frac{1}{n}}\phi_{p,n}^{\frac{1}{p}}}{(\frac{p-1}{n-p})^{\frac{1}{p}-1}}\right)\left(\int_0^\infty I_p\big(K_t(f)\big)\,dt\right)^{\frac{1}{p}}, \quad f \in C_0^\infty\big(\mathbb{R}^n\big), \tag{3.5}$$

is equivalent to, the geometric inequality

$$\big(V(K)\big)^{\frac{n-p}{pn}} \leqslant \big(2^{1-\frac{1}{p}}\sqrt{\pi}^{\frac{1}{p}}\phi_{p,n}^{\frac{1}{p}}\big)\big(I_p(K)\big)^{\frac{1}{p}}, \quad K \in \mathcal{K}_0^n. \tag{3.6}$$

The inequalities (3.5) *and* (3.6) *are true.*

(ii) *The analytic inequality*

$$\left(\int_0^\infty I_p\big(K_t(f)\big)\,dt\right)^{\frac{1}{p}} \leqslant \left(\frac{2^{\frac{1}{p}-1}}{\sqrt{\pi}^{\frac{1}{p}} n^{\frac{1}{p}} \omega_n^{\frac{1}{n}} \phi_{p,n}^{\frac{1}{p}}}\right)\|\nabla f\|_p, \quad f \in C_0^\infty\big(\mathbb{R}^n\big) \tag{3.7}$$

is true.

(iii) *The inequalities* (3.5) *and* (3.7) *are sharp and stronger than* (3.1) *for any* $f \in C_0^\infty(\mathbb{R}^n)$.

Proof. (i) Suppose (3.5) is true for all $f \in C_0^\infty(\mathbb{R}^n)$. Since

$$\int_0^\infty I_p\big(K_t(f)\big)\,dt \leqslant \left(\frac{n^{\frac{p}{n}}}{2^p}\right)\left(\int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{R}^n} \big|\langle v, \nabla f(x)\rangle\big|^p\,dx\right)^{-\frac{n}{p}} dv\right)^{-\frac{p}{n}}$$

holds for all $f \in C_0^\infty(\mathbb{R}^n)$—see also (ii) below, we conclude that

$$\|f\|^p_{\frac{pn}{n-p}} \leqslant \left(\frac{2^{-1}\sqrt{\pi}\phi_{p,n} n^{\frac{p}{n}} \psi_{p,n}^{\frac{p}{n}}}{(\frac{p-1}{n-p})^{1-p}}\right)\left(\int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{R}^n} \big|\langle v, \nabla f(x)\rangle\big|^p\,dx\right)^{-\frac{n}{p}} dv\right)^{-\frac{p}{n}}$$

is valid for all $f \in C_0^\infty(\mathbb{R}^n)$, and consequently, the last inequality keeps valid for the following function:

$$f_K(x) = \big(1 + \big(\max\{t \geqslant 0\colon\ tx \in K\}\big)^{\frac{p}{p-1}}\big)^{1-\frac{n}{p}}, \quad K \in \mathcal{K}_0^n,\ x \in \mathbb{R}^n.$$

Now, as estimated in [9, pp. 31–32 and Remark], we obtain a chain of inequalities and equalities

$$\begin{aligned}
&\psi_{p,n}^{\frac{p-n}{n}}\big(V(K)\big)^{\frac{n-p}{n}} \\
&\quad \leqslant \|f_K\|^p_{\frac{pn}{n-p}} \\
&\quad \leqslant \big(2^{-1}\sqrt{\pi} n^{\frac{p}{n}} \psi_{p,n}^{\frac{p}{n}} \phi_{p,n}\big)\left(\frac{p-1}{n-p}\right)^{p-1}\left(\int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{R}^n} \big|\langle u, \nabla f_K(x)\rangle\big|^p\,dx\right)^{-\frac{n}{p}} du\right)^{-\frac{p}{n}} \\
&\quad = \big(2^{p-1}\sqrt{\pi} n^{1+\frac{p}{n}} \psi_{p,n}^{\frac{p}{n}-1} \phi_{p,n}\big)\left(\int_{\mathbb{S}^{n-1}} \big(V_p(K,\bar{u})\big)^{-\frac{p}{n}} du\right)^{-\frac{p}{n}} \\
&\quad = \big(2^{p-1}\sqrt{\pi}\psi_{p,n}^{\frac{p}{n}-1}\phi_{p,n}\big) I_p(K),
\end{aligned}$$

whence deriving (3.6).

On the other hand, suppose (3.6) is true for any $K \in \mathcal{K}_0^n$. Given $f \in C_0^\infty(\mathbb{R}^n)$, we have that (3.6) is valid for $K_t(f) \in \mathcal{K}_0^n$ for almost every $t > 0$. Using the last five inequalities in [9, p. 31], we obtain

$$\begin{aligned}\|f\|^p_{\frac{pn}{n-p}} &\leqslant \psi^{\frac{p}{n}}_{p,n}\left(\frac{n-p}{p-1}\right)^{1-p}\int_0^\infty \big(V\big(K_t(f)\big)\big)^{\frac{n-p}{n}}\,dt\\ &\leqslant \big(2^{p-1}\sqrt{\pi}\,\psi^{\frac{p}{n}}_{p,n}\phi_{p,n}\big)\left(\frac{n-p}{p-1}\right)^{1-p}\int_0^\infty I_p\big(K_t(f)\big)\,dt,\end{aligned}$$

whence deriving (3.5).

Essentially, (3.6) for $K \in \mathcal{K}_0^n$ is just Lutwak–Yang–Zhang's L_p affine isoperimetric inequality in [9, (2.2)]. Accordingly, both (3.5) and (3.6) hold.

(ii) If $f \in C_0^\infty(\mathbb{R}^n)$, then an application of [9, (6.2) and (7.1)] and Minkowski's inequality yields

$$\begin{aligned}\int_0^\infty I_p\big(K_t(f)\big)\,dt &\leqslant \left(\frac{n^{\frac{p}{n}}}{2^p}\right)\left(\int_{\mathbb{S}^{n-1}}\left(\int_0^\infty V_p\big(K_t(f),\bar u\big)\,dt\right)^{-\frac{n}{p}}du\right)^{-\frac{p}{n}}\\ &= \left(\frac{n^{\frac{p}{n}}}{2^p}\right)\left(\int_{\mathbb{S}^{n-1}}\left(\int_0^\infty V_p(f,t,\bar u)\,dt\right)^{-\frac{n}{p}}du\right)^{-\frac{p}{n}}\\ &= \left(\frac{n^{\frac{p}{n}}}{2^p}\right)\left(\int_{\mathbb{S}^{n-1}}\left(\int_{\mathbb{R}^n}\big|\big\langle v,\nabla f(x)\big\rangle\big|^p\,dx\right)^{-\frac{n}{p}}dv\right)^{-\frac{p}{n}}\\ &\leqslant \left(\frac{(\frac{p-1}{n-p})^{p-1}}{2^{p-1}\sqrt{\pi}\,n\omega_n^{\frac{p}{n}}\phi_{p,n}}\right)\|\nabla f\|_p^p,\end{aligned}$$

as desired.

(iii) Since the equality in (3.6) occurs when and only when K is an ellipsoid centered at the origin—see also [9, Theorem 2.1] or [8, Theorem 2], we conclude that (3.5) is sharp, too. Also because (3.5) and (3.7) produce the sharp p-Sobolev inequality (3.1), we find out that (3.7) must be the best possible. □

Remark 3.4. Being connected with Theorem 3.3, we wonder:

(i) Whether the functional

$$\left(\int_0^\infty I_p\big(K_t(f)\big)\,dt\right)^{\frac{1}{p}}$$

in (3.5) and (3.7) may be replaced with the functional

$$\left(\int_0^\infty \big(I_p\big(K_t(f)\big)\big)^{\frac{n}{n-p}}\,dt^{\frac{pn}{n-p}}\right)^{\frac{n-p}{pn}}.$$

(ii) Whether a sharp geometric form (analogous with (1.12) but equivalent to (3.7)) can be found. In so doing, it seems necessary to define a p-surface area which can sharply dominate $I_p(\cdot)$ for $\mathcal{K}_0^n$.

Finally, we pose a problem rooted in Theorem 1.5.

Problem 3.5. Let $1 < p < n$. Then we conjecture:

(i) The analytic inequality

$$\int_0^\infty I_p\big(K_t(f)\big)\,dt \leqslant \left(\frac{(\frac{p-1}{n-p})^{p-1}}{2^{p-1}\sqrt{\pi}\,n\omega_n^{\frac{p}{n}}\,\phi_{p,n}}\right)\int_0^\infty C_p\big(K_t(f)\big)\,dt, \quad f \in C_0^\infty\big(\mathbb{R}^n\big), \tag{3.8}$$

is equivalent to, the geometric inequality

$$I_p(K) \leqslant \left(\frac{(\frac{p-1}{n-p})^{p-1}}{2^{p-1}\sqrt{\pi}\,n\omega_n^{\frac{p}{n}}\,\phi_{p,n}}\right)C_p(K), \quad K \in \mathcal{K}_0^n. \tag{3.9}$$

(ii) The inequalities (3.8) and (3.9) are true and sharp.

In order to prove or disprove this conjecture, we seemingly have to bring some new ideas (relative to [10,11]) into play. Note that the equality of (3.9) takes place when K is a closed ball of $\mathbb{R}^n$ centered at the origin. Furthermore, a combination of (3.9) and (3.6) yields (3.3) which is valid for any $K \in \mathcal{K}_0^n$. So, it is reasonable to expect the conjecture to be answered in the affirmative.

Acknowledgments

The author is grateful to G. Zhang for his help with various drafts of this paper. Also, it is author's pleasure to acknowledge alphabetically A. Colesanti, E. Lutwak, V. Mazýa and D. Yang for their comments on this work. Last but not least, the author would like to thank the referee for many suggestions improving the readability of the paper.

References

[1] T. Aubin, Problémes isopérimetriques et espaces de Sobolev, J. Differential Geom. 11 (1976) 573–598.
[2] H. Federer, W.H. Fleming, Normal and integral currents, Ann. of Math. 72 (1960) 458–520.
[3] W. Firey, p-Means of convex bodies, Math. Scand. 10 (1962) 17–24.
[4] R.J. Gardner, Geometric Tomography, Encyclopedia Math. Appl., vol. 58, Cambridge Univ. Press, Cambridge, 1995.
[5] R.J. Gardner, The Brunn–Minkowski inequality, Bull. Amer. Math. Soc. 39 (2002) 355–405.
[6] E. Lutwak, The Brunn–Minkowski–Firey theory I: Mixed volumes and the Minkowski problem, J. Differential Geom. 38 (1993) 131–150.
[7] E. Lutwak, The Brunn–Minkowski–Firey theory II: Affine and geominimal surface areas, Adv. Math. 118 (1996) 244–294.
[8] E. Lutwak, D. Yang, G. Zhang, L_p affine isoperimetric inequalities, J. Differential Geom. 56 (2000) 111–132.
[9] E. Lutwak, D. Yang, G. Zhang, Sharp affine L_p Sobolev inequalities, J. Differential Geom. 62 (2002) 17–38.
[10] E. Lutwak, D. Yang, G. Zhang, Volume inequalities for subspaces of L_p, J. Differential Geom. 68 (2004) 159–184.

[11] E. Lutwak, D. Yang, G. Zhang, Optimal Sobolev norms and the L_p Minkowski problem, Int. Math. Res. Not. 2006 (2006), Art. ID 62987, 21 pp.
[12] V. Maz'ya, Classes of regions and imbedding theorems for function spaces, Soviet Math. Dokl. 1 (1960) 882–885.
[13] V. Maz'ya, Sobolev Spaces, Springer-Verlag, Berlin, 1985.
[14] V. Maz'ya, Lectures on isoperimetric and isocapacitary inequalities in the theory of Sobolev spaces, Contemp. Math. 338 (2003) 307–340.
[15] V. Maz'ya, Conductor and capacitary inequalities for functions on topological spaces and their applications to Sobolev-type imbeddings, J. Funct. Anal. 224 (2005) 408–430.
[16] V. Maz'ya, T. Shaposhnikova, Theory of Multipliers in Spaces of Differentiable Functions, Monographs and Studies in Math., vol. 23, Pitman, 1985.
[17] R. Osserman, The isoperimetric inequality, Bull. Amer. Math. Soc. 84 (1978) 1182–1238.
[18] G. Rosen, Minimum value for C in the Sobolev inequality $\|\phi^3\| \leqslant C\|\operatorname{grad}\phi\|^3$, SIAM J. Appl. Math. 21 (1971) 30–33.
[19] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, Encyclopedia Math. Appl., vol. 44, Cambridge Univ. Press, Cambridge, 1993.
[20] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976) 353–372.
[21] A.C. Thompson, Minkowski Geometry, Encyclopedia Math. Appl., vol. 63, Cambridge Univ. Press, Cambridge, 1996.
[22] G. Zhang, The affine Sobolev inequality, J. Differential Geom. 53 (1999) 183–202.